(7) QN. → If $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$ then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

Ans. → we have, $u = \frac{y}{z} + \frac{z}{x} + \frac{x}{y}$

Diff. partially w.r.t. x, keeping y and z constant

$$\frac{\partial u}{\partial x} = 0 + z \times \frac{-1}{x^2} + \frac{1}{y} \times 1 = -\frac{z}{x^2} + \frac{1}{y}$$

$$x\frac{\partial u}{\partial x} = -\frac{z}{x} + \frac{x}{y} \quad \text{--- (1)}$$

Similarly $\frac{\partial u}{\partial y} = \frac{1}{z} + 0 + x \times \frac{-1}{y^2} = \frac{1}{z} - \frac{x}{y^2}$

$$y\frac{\partial u}{\partial y} = \frac{y}{z} - \frac{x}{y} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = y \times -\frac{1}{z^2} + \frac{1}{x} + 0$$

$$\frac{\partial u}{\partial z} = -\frac{y}{z^2} + \frac{1}{x}$$

$$z\frac{\partial u}{\partial z} = -\frac{y}{z} + \frac{z}{x} \quad \text{--- (3)}$$

(1) + (2) + (3)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -\frac{z}{x} + \frac{x}{y} + \frac{y}{z} - \frac{x}{y} - \frac{y}{z} + \frac{z}{x} = 0$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0. \text{ Proved}$$

(8) If $u = 3x^2yz + 5xy^2z + 4z^4$, Show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 4u.$$

Ans.→ Here, $u = 3x^2yz + 5xy^2z + 4z^4$

$$\frac{\partial u}{\partial x} = 6xyz + 5y^2z + 0$$

$$x\frac{\partial u}{\partial x} = 6x^2yz + 5xy^2z \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = 3x^2z + 5xz \times 2y + 0$$

$$\frac{\partial u}{\partial y} = 3x^2z + 10xyz$$

$$y\frac{\partial u}{\partial y} = 3x^2yz + 10xy^2z \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial z} = 3x^2y \times 1 + 5xy^2 + 16z^3$$

$$z\frac{\partial u}{\partial z} = 3x^2yz + 5xy^2z + 16z^4 \quad \text{--- (3)}$$

(1) + (2) + (3)

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=6x^2yz+5xy^2z+3x^2yz+$$
$$10xy^2z+3x^2yz+5xy^2z+16z^4$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=12x^2yz+20xy^2z+16z^4$$
$$=4\left[3x^2yz+5xy^2z+4z^4\right]$$
$$=4u \text{ proved.}$$

⑨ QN.→ If $u=x^2y+y^2z+z^2x$, Show that
$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=(x+y+z)^2$$

Ans.→ we have, $u=x^2y+y^2z+z^2x$

$$\therefore \frac{\partial u}{\partial x}=2xy+0+z^2=2xy+z^2$$

$$\frac{\partial u}{\partial y}=x^2+2yz+0=x^2+2yz$$

$$\frac{\partial u}{\partial z}=0+y^2+2zx=y^2+2zx$$

$$\therefore L.H.S=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$$

$$=2xy+z^2+x^2+2yz+y^2+2zx$$
$$=x^2+y^2+z^2+2xy+2yz+2zx$$
$$=(x+y+z)^2 \text{ Proved}$$

⑩ If $u=(x^2+y^2+z^2)^{-\frac{1}{2}}$, Show that
$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=-u.$$

Ans. → (i) $\because u = (x^2+y^2+z^2)^{-\frac{1}{2}}$

$$\therefore \frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}} \cdot 2x = -x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\text{i.e. } x\frac{\partial u}{\partial x} = -x^2(x^2+y^2+z^2)^{-\frac{3}{2}}$$

Similarly. $$y\frac{\partial u}{\partial y} = -y^2(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$z\frac{\partial u}{\partial z} = -z^2(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -x^2(x^2+y^2+z^2)^{-\frac{3}{2}} + -y^2(x^2+y^2+z^2)^{-\frac{3}{2}} + -z^2(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -(x^2+y^2+z^2)^{-\frac{3}{2}}(x^2+y^2+z^2)$$

$$= -(x^2+y^2+z^2)^{-\frac{1}{2}}$$

$$= -u.$$

(11) QN. → If $u = \log(\tan x + \tan y + \tan z)$, show that

$$\sin 2x\frac{\partial u}{\partial x} + \sin 2y\frac{\partial u}{\partial y} + \sin 2z\frac{\partial u}{\partial z} = 2.$$

Ans. → $\because u = \log(\tan x + \tan y + \tan z)$

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \times \sec^2 x,$$

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z}$$

$$\frac{\partial u}{\partial z} = \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\sin 2x \frac{\partial u}{\partial z} = \frac{\sin 2x \times \sec^2 x}{\tan x + \tan y + \tan z} = \frac{2 \sin x \cdot \cos x}{(\tan x + \tan y + \tan z) \times \cos^2 x}$$

$$= \frac{2 \tan x}{\tan x + \tan y + \tan z} \quad \text{— (1)}$$

Similarly $\sin 2y \frac{\partial u}{\partial y} = \frac{2 \tan y}{\tan x + \tan y + \tan z}$ — (2)

and $\sin 2z \frac{\partial u}{\partial z} = \frac{2 \tan z}{\tan x + \tan y + \tan z}$ — (3)

(1) + (2) + (3)

$$\therefore \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = \frac{2(\tan x + \cancel{\tan y} + \tan z)}{\tan x + \cancel{\tan y} + \tan z}$$

$= 2$ Proved.

(12) QN. → If $u = 2(ax + by)^2 - (x^2 + y^2)$ and $a^2 + b^2 = 1$,

prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Ans. → $\because u = 2(ax + by)^2 - (x^2 + y^2)$

$$\frac{\partial u}{\partial x} = 4(ax + by) \times a - 2x$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\{4a(ax + by) - 2a\} = 4a(a + 0) - 2$$

$$= 4a^2 - 2 \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = 4(ax+by) \times b - 2y$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\{4b(ax+by) - 2y\} = 4b\{b-2\} = 4b^2 - 2 \quad -(2)$$

(1) + (2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4a^2 - 2 + 4b^2 - 2$$

$$= 4(a^2+b^2) - 4$$

$$= 4 \times 1 - 4 = 4 - 4 = 0 = R.H.S.$$

(13) If $u = \tan^{-1}\frac{y}{x}$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

Ans.→ $\because u = \tan^{-1}\frac{y}{x}$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times y \times -\frac{1}{x^2}$$

$$= -\frac{\cancel{x^2}}{x^2+y^2} \cdot \frac{y}{\cancel{x^2}} = -\frac{y}{x^2+y^2}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right)$$

$$= -y\frac{\partial}{\partial x}(x^2+y^2)^{-1}$$

$$= -y \times -1 \times (x^2+y^2)^{-2} \times 2x$$

$$\frac{\partial^2 u}{\partial x^2} = +\frac{2yx}{(x^2+y^2)^2} \quad -(1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x} \times 1$$

$$\frac{\partial u}{\partial y} = \frac{x^2}{x^2+y^2} \times \frac{1}{x} = x(x^2+y^2)^{-1}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = x\frac{\partial}{\partial u}(x^2+y^2)^{-1}$$

$$= x \times -1 (x^2+y^2)^{-2} \times 2y$$

$$= \frac{-2xy}{(x^2+y^2)^2} \quad -(2)$$

(1) + (2)

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} = \cancel{\frac{2xy}{(x^2+y^2)^2}} - \cancel{\frac{2xy}{(x^2+y^2)^2}} = 0 \text{ Pro}$$

(14) QN. → If $u = \frac{\sin(ct-x)}{x}$, prove that

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 x}{\partial x^2} + \frac{2}{x}\cdot\frac{\partial u}{\partial x}\right).$$

Ans. → we have $u = \frac{\sin(ct-x)}{x}$

~~Diff. partially w.r.t. t,~~

$$\therefore \frac{\partial u}{\partial t} = \frac{\cos(ct-x)}{x} \times c$$

and $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{-c \times c}{x} \times \sin(ct-x)$$

$$= \frac{-c^2}{x}\cdot\sin(ct-x) \quad -(1)$$

Also $$\frac{\partial u}{\partial x} = -\frac{1}{x^2}\sin(ct-x) - \frac{1}{x}\cos(ct-x) \quad (2)$$

and $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{1}{x^2}\cdot\cos(ct - x) + \frac{1}{x^2} + \frac{1}{x}$

$(\sin(ct - x), + \frac{1}{x^2}\cos(ct - x) - \frac{1}{x}$

$\sin(ct - x)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{x^2}\cos(ct - x) + \frac{2}{x^3}\sin(ct - x) + \frac{1}{x^2}\cos(ct-x)$$

$$-\frac{1}{x}\sin(ct - x) \quad -(3)$$

(2) + (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\cdot\frac{\partial u}{\partial x} = -\frac{1}{x}\sin(ct - x) = \frac{1}{c^2}\cdot\frac{\partial^2 u}{\partial t^2}$$

$$\therefore c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{2}{x}\cdot\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial t^2}$$

(22) QN. → If ∇^2 stands for the operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ prove that ~~where, $\frac{1}{r}$ =~~

$\nabla^2 r = 0$

where, $\frac{1}{r^2} = x^2 + y^2 + z^2$.

or,

If $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$. Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

Ans. → $\because \frac{1}{r^2} = x^2 + y^2 + z^2$

$$\therefore \frac{-2}{r^3}\cdot\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = -x r^3$$

$$\frac{\partial r}{\partial x} = -r^3 x$$

similarly $\frac{\partial r}{\partial y} = r^3 y$ (a)

$$\frac{\partial r}{\partial z} = -r^3 z$$

Again diff. (a) partially w.r.t. x

$$\frac{\partial^2 r}{\partial x^2} = -r^3 \times 1 + x \times -3r^2 \frac{\partial r}{\partial x}$$

$$= -r^3 - 3xr^2 \times -r^3 x$$

$$\frac{\partial^2 r}{\partial x^2} = -r^3 + 3x^2 r^5 \quad \text{---(1)}$$

similarly $\frac{\partial^2 u}{\partial y^2} = -r^3 + 3y^2 r^5$ ---(2)

$$\frac{\partial^2 u}{\partial z^2} = -r^3 + 3z^2 r^5 \quad \text{---(3)}$$

(1) + (2) + (3)

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = -3r^3 + 3r^5(x^2+y^2+z^2)$$

$$= -3r^3 + 3r^5 \times \frac{1}{r^2}$$

$$= -\cancel{3r^3} + \cancel{3r^3}$$

$= 0$ proved.